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2002 J. Phys. A: Math. Gen. 35 8913

(http://iopscience.iop.org/0305-4470/35/42/303)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8913–8928 PII: S0305-4470(02)37996-4

Special set and solutions of coupled nonlinear schrödinger equations

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Received 10 June 2002
Published 8 October 2002
Online at stacks.iop.org/JPhysA/35/8913

Abstract
A special set of N-coupled nonlinear Schrödinger equations which consists of 2^N interaction types each of which is characterized by a specific array of interaction parameters is presented. It is shown that *every* Lamé function of order $n \leqslant N$ is a solution for one or more components for one or more interaction types of this special set. Simple rules that relate interaction and solution types and interesting features of these relationships are presented.

PACS numbers: 05.45.Yv, 02.30.Jr, 02.30.Ik

1. Introduction

Coupled nonlinear Schrödinger (CNLS) equations involving N components, because of their importance in many physical and mathematical problems, have been studied extensively for many years [1]. The complex amplitude $\phi_m(z, t)$ of the mth component as a function of position z and time t is assumed to satisfy the following N CNLS equations:

$$\mathrm{i}\phi_{mz} + \varepsilon_m \phi_{mtt} + \kappa_m \phi_m + \left(\sum_{j=1}^{N} \lambda_{mj} |\phi_j|^2 \right) \phi_m = 0 \qquad m = 1, \ldots, N \tag{1}$$

where ε_m, κ_m and λ_{mj} are real parameters characteristic of the medium and interaction, and where the subscripts in z and t denote derivatives with respect to z and t, and the subscript m is for different components.

Soliton [1–3] and solitary-wave solutions [3–5], Painlevé analysis and integrability [6], of CNLS equations have been presented for special sets of parameters ε_m, κ_m and λ_{mj}. In particular, Lakshmanan and his collaborators [6] have identified a specific set of parameters that possess the Painlevé property, where the set of interaction parameters λ_{mj} can consist of a specific mixture of negative as well as positive real constants of equal magnitude. They have also obtained soliton solutions for $N = 2$ and 3 for those 'mixed' cases in addition to the bright and dark soliton solutions.

In this paper, we show that there is a special set that consists of 2^N specific arrays of interaction parameters for which the N CNLS equations possess special analytic solutions. In these analytic solutions, the N components of the CNLS equations are expressed in terms of N Lamé functions [7] and every Lamé functions of order $n \leqslant N$ is a solution for one or more components for one or more interaction types of this special set. Explicit solutions for various specific cases of interaction parameters for $n = N = 1$–4 are presented. A collection of N Lamé functions that can serve as an analytic solution for the N components of these CNLS equations will be referred to as a combination. In this paper, we present simple rules that (A) identify a given combination to be a solution to one or more specific interaction types and (B) give all the possible combinations as solutions of a specific interaction type. These 2^N specific sets of CNLS equations that have these Lamé functions as special solutions will be called the L-set. It is significant to note that the L-set we found coincides with the set of CNLS equations that pass the Painlevé test identified by Radhakrishnan *et al* [6]. Potentially wide ranging physical applications of our results are mentioned in the summary.

2. Special analytic solutions

The special analytic solutions we shall present apply to specific parameters appearing in the N CNLS equations (1). We assume that the ε_m in equation (1) all have the same magnitude but possibly different signs, and we normalize them to be equal to +1 or −1. We also assume that the λ_{mj} all have equal magnitude but possibly different signs, and we also normalize them to be equal to +1 or −1. In most physical applications, the $N \times N$ matrix for λ_{mj} is symmetric, i.e. $\lambda_{mj} = \lambda_{jm}$, and this is the case we shall study. We first search for the stationary-wave solution of the form

$$\phi_m(z, t) = \psi_m(t) \exp(\mathrm{i}\omega z) \tag{2}$$

where ω is a real constant and $\psi_m(t)$ are real functions of t only. Although ω can depend on m and thus should be written as ω_m for stationary wave, for the purpose of constructing a propagating-wave solution from the stationary-wave solution described below, ω is assumed to be independent of m. We consider the following N equations for $\psi_m(t)$, which may be referred to as the 'dynamical' CNLS equations

$$\psi_{mtt} + c_m\psi_m + \left(\sum_{j=1}^{N} \beta_j \psi_j^2\right) \psi_m = 0 \qquad m = 1, \ldots, N \tag{3}$$

where $\beta_j = +1$ or -1, to come from subsets of equation (1) given by

$$\mathrm{i}\phi_{mz} \pm \beta_m\phi_{mtt} + \kappa_m\phi_m \pm \left(\sum_{j=1}^{N} \beta_m\beta_j|\phi_j|^2\right) \phi_m = 0 \qquad m = 1, \ldots, N \tag{4}$$

where

$$c_m = \pm\beta_m(\kappa_m - \omega) \tag{5}$$

and all quantities in equation (3) are assumed real. To eliminate the permutation symmetry, we arrange equation (3) such that

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N \tag{6}$$

so that only one of the two choices (the upper or lower sign) in equations (4) and (5) corresponds to the equations of motions for equation (3). The travelling waves, if required, can be constructed by substituting the solutions ψ_m from equation (3) into equation (2), and replacing

$\phi_m(z, t)$ by $\phi_m(z, t - z/v) \exp\{i[t - z/(2v)]/(2v)\}$, where v is the common velocity of the waves.

We now characterize the interaction parameters of equation (4) by the array $(\beta_1, \beta_2, \ldots, \beta_N)$, where $\beta_j = +1$ or -1, and refer to each of the 2^N arrays as an interaction type. Note that the 2^N interaction types constitute only a subset of equation (1) for symmetric λ_{mn} that can take on values +1 or -1, because the symmetric λ_{mn} are now restricted to be those given by $\lambda_{mn} = \beta_m\beta_n$ (or $-\beta_m\beta_n$). As we shall see, however, this is an interesting and important subset which we shall refer to as the L-set. Our analytic and numerical computations suggest that this special set of CNLS equations represented by equation (4) possesses special analytic solutions for $\phi_m(z, t)$ that can be expressed in terms of Lamé functions of order $n \leqslant N$. Our results also suggest that any N-combination of Lamé functions of order $n \leqslant N$ is a solution of one or more of the interaction types belonging to the L-set, and we have found a simple rule that identifies a combination as solutions of certain interaction types and vice versa.

3. Combinations for given interactions

Lamé equation of order n can be written in the form [7]

$$d^2 f/d\tau^2 + [h - n(n+1)k^2 \operatorname{sn}^2(\tau, k)]f = 0. \tag{7}$$

We shall use only the polynomial solutions of the Lamé equation and refer to them as Lamé functions, and we shall number the $2n + 1$ Lamé functions of order n, $f_1^{(n)}, f_2^{(n)}, \ldots, f_{2n+1}^{(n)}$, in the order of numbering their corresponding eigenvalues $h_m^{(n)}$ arranged in descending order $h_1^{(n)} > h_2^{(n)} > \cdots > h_{2n+1}^{(n)}$. Lamé functions of order n for $n = 1$–5 and their corresponding eigenvalues are given in appendix A. A crucial step in our analysis is the use of the Lamé function ansatz [4] generalized to a wider variety of interaction parameters for N CNLS equations. While it is known that the N components of CNLS equations in which the interaction parameters are equal and have the same sign have analytic solutions expressible in terms of *specific* combinations of N Lamé functions of order $n \leqslant N$ [5, 8], consideration of a wider variety of interaction parameters [4] led us to the L-set which allows *all* possible combinations of Lamé functions of order $n \leqslant N$ as solutions, as we shall present in this paper.

The use of Lamé function ansatz described in [4] gives the solutions of equation (3) in terms of Lamé functions in the form

$$\psi_m(t) = C_m f_p^{(n)}(t) \tag{8}$$

where C_m is the (real) 'amplitude' of the mth component and $f_p^{(n)}$ is one of the $2n + 1$ Lamé functions of order n. The solution requires specific values for the c_m of equation (3) and thus κ_m of equation (4). An N-combination $\left(f_p^{(n)}, f_q^{(n)}, \ldots, f_s^{(n)}\right)$ that gives an analytic solution for the N components $(\psi_1, \psi_2, \ldots, \psi_N)$ will be represented simply by $(p, q, \ldots, s)_n$, where equation (6) implies $p \leqslant q \leqslant \cdots \leqslant s$. We first renumber the $2n + 1$ eigenvalues of the Lamé equation $h_1^{(n)}, h_2^{(n)}, \ldots, h_{2n+1}^{(n)}$, as $h_1^{(n)}, h_2^{(n)}, h_{2'}^{(n)}, \ldots, h_{n+1}^{(n)}, h_{(n+1)'}^{(n)}$, and the corresponding Lamé functions $f_1^{(n)}, f_2^{(n)}, \ldots, f_{2n+1}^{(n)}$, as $f_1^{(n)}, f_2^{(n)}, f_{2'}^{(n)}, \ldots, f_{n+1}^{(n)}, f_{(n+1)'}^{(n)}$, i.e. we group them in pairs except the first one. An N-combination $\left(f_p^{(n)}, f_q^{(n)}, f_{r'}^{(n)}, \ldots, f_s^{(n)}\right)$, for example, will be represented by $(p, q, r', \ldots, s)_n$, where $p \leqslant q \leqslant r' \leqslant \cdots \leqslant s$.

To obtain the solutions of equation (3) using the Lamé function ansatz [4], we express the square of the jth Lamé function of order n in a power series in $s \equiv \operatorname{sn}(\tau, k)$ as

$$\left[f_j^{(n)}(\tau)\right]^2 = \sum_{i=1}^{n+1} a_{ij}^{(n)} s^{2(i-1)} \qquad j = 1, \ldots, 2n+1$$

and assume that $\psi_1, \ldots, \psi_N$ are expressible as $\psi_1(t) = C_1 f_p^{(n)}(\alpha t), \psi_2(t) = C_2 f_q^{(n)}(\alpha t), \ldots, \psi_N(t) = C_N f_s^{(n)}(\alpha t)$. Substituting these into equation (3) and comparing them with equation (7) give a set of algebraic equations that need to be satisfied. Solving the equations gives the required values for the amplitudes C and the required values of c for equation (3). Using the analytic expressions for the Lamé functions presented in appendix A for $N = 1$–5, all possible analytic solutions of equation (3) can be obtained. As the coefficients in the expansions for the Lamé functions in powers of s can be obtained by a recursive procedure, the method can be efficiently applied numerically to any value of N.

Analytic solutions involving specific combinations for specific interaction types given by $(\beta_1, \beta_2, \ldots, \beta_N) = (+, +, \ldots, +), (+, -, +, \ldots, -)$ and $(-, +, -, \ldots, -)$ are presented in appendix B for the special case of $N = n$ for $N = 1$–4. We have also grouped all possible $M = \binom{2n+1}{N}$ combinations of Lamé functions of order $n = N$ that are solutions to the 2^N interaction types for $N = 1$–5 and present them in appendix C.

The main results of this paper concern the question of what combinations (of Lamé functions) are applicable to what interaction types. Based on our above results for $n \leqslant N$, we summarize the principal features and make the following assertions for a general N. We first distinguish two separate cases of representing the solutions in terms of Lamé functions: case (I) is in terms of Lamé functions of order $n = N$ and case (II) in terms of Lamé functions of order $n < N$. We have the following:

(I) For $n = N$, the N Lamé functions for the N components must necessarily be different Lamé functions. The number of N combinations that can be chosen from $2n + 1$ distinct Lamé functions of order $n = N$, with no repetition allowed, is $M = \binom{2n+1}{N}$.

Every one of the M combination is a solution of one and only one particular interaction type, and every interaction type has one or more combinations as solutions. More specifically,

(A) combination $(p_1, p_2 + 1, p_3 + 2, \ldots, p_N + N - 1)_n$ is a solution of interaction type $((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3}, \ldots, (-1)^{p_N})$, and

(B) an interaction type $((-1)^{p_1}, (-1)^{p_2}, (-1)^{p_3}, \ldots, (-1)^{p_N})$ has solutions given by all possible combinations $(m_1, m_2, m_3, \ldots, m_N)_n$, which can be obtained by setting $m_1 = p_1, m_2 = p_2 + 1, m_3 = p_3 + 2, \ldots, m_N = p_N + N - 1$, and by increasing or decreasing m_j by any multiple of 2 subject to the condition that $m_1 < m_2 < m_3 < \cdots < m_N$, where m_j can take on values $1, 2, 2', 3, 3', \ldots, n + 1, (n + 1)'$ with the understanding that $r < r'$.

Using this rule, the grouping of the $M = \binom{2n+1}{N}$ combinations for the 2^N interaction types for the L-set for $N = 1$–5, which are explicitly shown in appendix C can be verified. Note that *every one* of the M possible combinations is accounted for and appears in the list for each N only once, i.e. every combination is a solution of one interaction type of the L-set. It can be seen from the rule given above the following special cases:

(a) The interaction type $(- - - \cdots -)$ allows 2^{N-1} combinations as solutions given by $(1, 2, 3, \ldots, N)_n$ and those in which one or more of the $N - 1$ numbers $2, 3, \ldots, N$ are replaced by $2', 3', \ldots, N'$;

(b) The interaction type $(+ + + \cdots +)$ allows 2^N combinations as solutions given by $(2, 3, \ldots, N + 1)_n$ and those in which one or more of the N numbers $2, 3, \ldots, N + 1$ are replaced by $2', 3', \ldots, (N + 1)'$;

(c) For N even, the interaction type $(+ - + - \cdots + -)$ allows only one combination $(22'44' \cdots NN')_n$ as solution; and the interaction type $(- + - + \cdots - +)$ allows $N + 1$ combinations as solutions given by $(3, 3', 5, 5', \ldots N + 1, (N + 1)')_n$ and N other combinations obtained by replacing one of the N numbers by 1.

(d) For N odd, the interaction type $(- + - + \cdots -)$ allows only one combination $(133'55' \cdots NN')_n$ as solution; and the interaction type $(+ - + - \cdots +)$ allows $N + 1$ combinations as solutions obtained from choosing N numbers from the $N+1$ numbers $2, 2', 4, 4', \ldots, N + 1, (N + 1)'$.

(II) For $n < N$, two or more of the Lamé functions for the N components may be the same function. The number of N combinations that can be chosen from $2n + 1$ distinct Lamé functions of order $n < N$ each of which may appear from 0 to N times is $M' = \binom{2n+1}{N}$.

A total of M' combinations $(m_1, m_2, \ldots, m_N)_n$ are possible where $m_1 \leqslant m_2 \leqslant m_3 \leqslant \cdots \leqslant m_N$ can be chosen, with repetition allowed, from $1, 2, 2', 3, 3', \ldots, n + 1, (n + 1)'$, and in general, these combinations, with a few exceptions, are possible solutions for each of the 2^N interaction types. By studying the equations derived from the Lamé function ansatz and the forms of Lamé functions, we find the following general restrictions:

(A) a combination $(m_1, m_2, \ldots, m_N)_n$ must have at least n distinct m for it to be a solution; and

(B) combination $(m_1, m_2, \ldots, m_N)_n$ is disallowed for interaction type $((-1)^{m_1+n}, (-1)^{m_2+n}, \ldots, (-1)^{m_N+n})$.

Unlike the case for $n = N$ for which the C and c must be specific values (see appendix B) for a combination to be a solution, for the case $n < N$, there can be one or more 'relatively' free parameter (which may be subject to a certain constraint) on which the C and c depend.

We present the disallowed combinations from the above rules (A) and (B) for $N = 2$, $n = 1$, and for $N = 3, n = 1$ and 2, as examples, in appendix D, as those for higher values of N can be easily written.

There are, however, other restrictions specific to certain kinds of combinations and interaction types. Some of these restrictions are quite apparent when we discuss the degenerate combinations in the next section for which an example is given in appendix E.

4. Degenerate and degenerative combinations

For the case of $N > n$ for which combinations with repetitions are allowed, we shall refer to a combination in which two or more of the m in the combination $(m_1, m_2, \ldots, m_N)_n$ are equal as a degenerate combination, and one in which all the m are distinct as a non-degenerate combination. For a degenerate combination, if we replace all the equal m by a single member, then we get what we shall call the corresponding 'contracted' (non-degenerate) combination. Using the Lamé function ansatz given in [4], it is easy to show that the amplitudes C_m for ψ_m and the required c_m for equation (3) for a degenerate combination can be obtained from the corresponding quantities for the corresponding contracted (and non-degenerate) combination involving N' distinct Lamé functions of order n. A simple rule for doing this is illustrated in appendix E.

For the case of $n = N$, all combinations must be non-degenerate (for $0 < k^2 < 1$) to be solutions of N CNLS equations (4). It is useful to divide the $M = \binom{2n+1}{N}$ combinations of Lamé functions that can be obtained from $2n + 1$ Lamé functions of order n (with no repetition allowed) into two kinds: the degenerative combination is one in which at least two of the m in the combination $(m_1, m_2, \ldots, m_N)_n$ involve a pair p and p', where p is one of the numbers from 2 to $n + 1$; the non-degenerative combination is one that is not degenerative. It can be shown that of the M combinations, the number of degenerative combinations is $M_d = n(2n-1)!/\{(N-2)!(2n-N+1)!\}$ and the number of non-degenerative combinations is $M_{nd} = M - M_d$.

We also divide the 2^N interaction types into two kinds: the kind in which all the – precede the + will be called the 'weakly' mixed type, which includes the 'pure' type in which the signs are all $-$ or all +; the other kind in which the $-$ appear before and after the + will be called the 'strongly' mixed type.

We find that the non-degenerative combinations are solutions of only the weakly mixed interaction type. The degenerative combinations can be solutions of the strongly mixed as well as the weakly mixed interaction types except the pure type. Division into degenerative and non-degenerative combinations aside, we again point out that every one of the M possible combinations (for $n = N$) of Lamé functions is a solution to one of the 2^N interaction types.

The division of combinations into degenerative and non-degenerative is consistent with consideration of Lamé functions as k^2 becomes equal to 1. In that case, the $2n+1$ eigenvalues, except for the first one, become pairwise degenerate, i.e. $h_2^{(n)} = h_{2'}^{(n)}, h_3^{(n)} = h_{3'}^{(n)}, \ldots, h_{(n+1)}^{(n)} = h_{(n+1)'}^{(n)}$, and the corresponding Lamé functions of order n become $n+1$ associated Legendre functions $P_n^m(x)$, $m = 0, 1, 2, \ldots, n$, where $x = \tanh(\alpha t)$ using α and t used in appendix B.

We now consider the $N+1$ weakly mixed interaction types and the corresponding solutions given by the non-degenerative combinations for the special case $k^2 = 1$. In this case, we find that the solutions, including the amplitudes C_m for equation (8) and the corresponding c_m required for equation (3) can be written in a compact form.

We first define our normalized associated Legendre functions $P_n^m(x)$ as follows:

$$P_n^m(x) = (1 - x^2)^{m/2}\,\mathrm{d}^m P_n(x)/\mathrm{d}x^m$$

where $P_n(x) = (2^{-n}/n!)\,\mathrm{d}^n(x^2-1)^n/\mathrm{d}x^n$.

To express our results in a compact way, we consider $(N+1)$ CNLS equations involving ψ_j, $j = 1, 2, \ldots, N+1$ with $j = r$ missing (or $\psi_r = 0$), where r can be any one of the $N+1$ components. Written explicitly, we have the following N dynamical CNLS equations:

$$\psi_{mtt} + c_m\psi_m + \left(\sum_{j=1}^{r-1} -\psi_j^2 + \sum_{j=r+1}^{N+1} \psi_j^2\right)\psi_m = 0 \qquad m = 1, \ldots, N+1. \tag{9}$$

We find the analytic solutions ψ_j with amplitudes C_j and the required c_j given by

$$\begin{aligned}
&\psi_j = C_j P_N^{j-1}(\tanh\alpha t) \qquad j = 1, \ldots, N+1 \quad j \neq r\\
&C_1 = [2(r-1)^2]^{1/2}\alpha\\
&C_j = \{4[(N-j+1)!]|(r-1)^2 - (j-1)^2|/(N+j-1)!\}^{1/2}\alpha\\
&c_j = [2(r-1)^2 - (j-1)^2]\alpha^2 \qquad j = 1, \ldots, N+1 \quad j \neq r
\end{aligned} \tag{10}$$

where the specification $j \neq r$ is not really necessary since setting $j = r$ would give C_j and c_j equal to zero anyway. However, the specification $j \neq r$ is a reminder that there are N and not $N+1$ coupled components of CNLS equations considered.

It can be checked that for $r = 1$, i.e. when the nonlinear coupling parameters in equation (9) are all equal to +1, the above results coincide with those given previously in [4, 5], and for $r = N+1$, i.e. when the nonlinear coupling parameters are all equal to -1, the above results coincide with those given previously in [4]. Thus our above results given by equation (9) generalize the previous results to the weakly mixed cases where the first $r-1$ of the β in equation (3) are equal to -1, and the remaining β are equal to +1. It may be noted that the 'generalized' dark solitary wave is $P_n^0(\tanh\alpha t)$ and the 'generalized' bright solitary wave is $P_n^n(\tanh\alpha t)$; they become the familiar dark solitary wave $\tanh\alpha t$ and the familiar bright solitary wave $\mathrm{sech}\,\alpha t$, respectively, for $n = 1$. The generalized dark solitary waves $P_n^0(\tanh\alpha t)$, unlike the rest of the set $P_n^m(\tanh\alpha t)$, $m = 1, \ldots, n$, do not become zero when $t \to \pm\infty$.

5. Comparison with *P*-test characterization

As mentioned in the introduction, it is significant that our L-set defined in section 2 coincides with the set of CNLS equations that pass the Painlevé test identified by Radhakrishnan *et al* [6]. While the 2^{N-1} specific parameter restrictions given by Radhakrishnan *et al* [6] (see their equations $(61a)$–$(61n+1)$) appear to be somewhat complicated compared to our characterization, it is instructive and suffices for our purpose to compare their and our characterizations for $N = 3$ explicitly. Using equation (4) and choosing the upper sign (there is no new physics if the lower sign is chosen) as our L-set for the general CNLS equations (1) means that our L-set is characterized by the column vector $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\beta_1, \beta_2, \beta_3)$ and the 3×3 matrix $\lambda_{mj} = \beta_m \beta_j$, $m, j = 1, 2, 3$, $\beta_j = +1$ or -1. It gives $2^N = 8$ interaction types, each with a column vector $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and the corresponding matrix $[\lambda_{mj}]$ as follows (we shall write + for +1 and − for −1 for convenience):

$N = 3 \qquad c_1 > c_2 > c_3$

$$(1)\ \begin{bmatrix} - \\ - \\ - \end{bmatrix} \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix} \qquad (2)\ \begin{bmatrix} - \\ - \\ + \end{bmatrix} \begin{bmatrix} + & + & - \\ + & + & - \\ - & - & + \end{bmatrix}$$

$$(3)\ \begin{bmatrix} - \\ + \\ - \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad (4)\ \begin{bmatrix} - \\ + \\ + \end{bmatrix} \begin{bmatrix} + & - & - \\ - & + & + \\ - & + & + \end{bmatrix}$$

$$(5)\ \begin{bmatrix} + \\ - \\ - \end{bmatrix} \begin{bmatrix} + & - & - \\ - & + & + \\ - & + & + \end{bmatrix} \qquad (6)\ \begin{bmatrix} + \\ - \\ + \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$(7)\ \begin{bmatrix} + \\ + \\ - \end{bmatrix} \begin{bmatrix} + & + & - \\ + & + & - \\ - & - & + \end{bmatrix} \qquad (8)\ \begin{bmatrix} + \\ + \\ + \end{bmatrix} \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}.$$

It is seen that (1) and (8), (2) and (7), (3) and (6), (4) and (5) are related by reversing the signs of the column vector $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ but having the same matrix $[\lambda_{mj}]$, and the pair is counted as one in the characterization given by Radhakrishnan *et al*, and thus giving $2^{N-1} = 4$ parametric restrictions that pass the *P*-test, as can be verified using the scheme of characterization given by them. Thus their and our characterizations give the same set of CNLS equations: theirs for the set that passes the *P*-test, and ours for the set that possesses the Lamé functions as analytic solutions. However, our characterization of the set not only is much simpler to state and remember, but also separates out different solution types.

6. Summary

We have presented the following assertions and results for N CNLS equations belonging to the L-set, i.e. for the special set of N CNLS equations given by equation (4):

(1) that every one of the M possible N combinations of distinct Lamé functions of order $n = N$ is a solution of one, and only one, of the 2^N interaction types;
(2) that every one of the M' possible N combinations of Lamé functions of order $n < N$ in which the same Lamé function may represent more than one component, is a solution of one or more of the 2^N interaction types;
(3) rules for identifying combinations with interaction types and for disallowed combinations;

(4) the relationship between degenerative, non-degenerative combinations and strongly and weakly mixed interaction types; and
(5) a compact general analytic expression for the N components ψ_j corresponding to the weakly mixed interaction types for the case $k^2 = 1$.

This special set of N CNLS equations coincides with the set of CNLS equations that pass the Painlevé test [6] even though the characterization of parameters was done differently in [6]. It can be seen by comparison, however, that our characterization of the interaction parameters is simpler, and that our characterization of the solution types gives a greater clarification to the nature of the interaction parameters.

As a consequence of this characterization, physical applications of our results are more apparent. In particular, (a) our results provide the analytic forms of solitary waves that can be made into partners (i.e. combination of Lamé functions) in order to propagate unattenuated through a medium of certain characteristics (i.e. interaction type such as a normal or an anomalous group-velocity dispersion region [1]) which each wave may not be able to do individually, and (b) as presented in a recent communication [9], some of our results provide the analytic forms of stationary distributions of coupled Bose–Einstein condensates and suggest possible ways of making them overlap each other (i.e. same Lamé functions appearing in the combination) or separated (i.e. different Lamé functions in the combination). Physical applications of our results for the mixed interaction types, which make up the L-set and are clearly important mathematically, still await us, and we believe, will come in the very near future and are potentially wide ranging.

Acknowledgment

This research is supported by NSF grant PHY-9900659.

Appendix A

The $2n+1$ Lamé functions $f_m^{(n)}(\tau)$ and their eigenvalues $h_m^{(n)}$ satisfy the Lamé equations (7), and we list them according to the subscript $m = 1, 2, 2', 3, 3', \ldots, n+1, (n+1)'$ described in section 3, for $n = 1$–5, in this appendix.

- $n = 1$

$$h_1 = 1 + k^2 \qquad h_2 = 1 \qquad h_{2'} = k^2$$
$$f_1 = \mathrm{sn}(\tau) \qquad f_2 = \mathrm{cn}(\tau) \qquad f_{2'} = \mathrm{dn}(\tau).$$

- $n = 2$

$$h_{1,3'} = 2(1+k^2) \pm 2\sqrt{1-k^2+k^4}$$
$$h_2 = 4 + k^2$$
$$h_{2'} = 1 + 4k^2$$
$$h_3 = 1 + k^2$$
$$f_{1,3'} = 1 - \left\{1 + k^2 \pm \sqrt{1-k^2+k^4}\right\} \mathrm{sn}^2(\tau)$$
$$f_2 = \mathrm{sn}(\tau)\,\mathrm{cn}(\tau)$$
$$f_{2'} = \mathrm{sn}(\tau)\,\mathrm{dn}(\tau)$$
$$f_3 = \mathrm{cn}(\tau)\,\mathrm{dn}(\tau).$$

- $n = 3$

$$h_{1,3'} = 5(1+k^2) \pm 2\sqrt{4-7k^2+4k^4}$$
$$h_{2,4} = 5+2k^2 \pm 2\sqrt{4-k^2+k^4}$$
$$h_{2',4'} = 2+5k^2 \pm 2\sqrt{1-k^2+4k^4}$$
$$h_3 = 4(1+k^2)$$
$$f_{1,3'} = \operatorname{sn}(\tau)\left\{1-\tfrac{1}{3}\left[2+2k^2 \pm \sqrt{4-7k^2+4k^4}\right]\operatorname{sn}^2(\tau)\right\}$$
$$f_{2,4} = \operatorname{cn}(\tau)\left\{1-\left[2+k^2 \pm \sqrt{4-k^2+k^4}\right]\operatorname{sn}^2(\tau)\right\}$$
$$f_{2',4'} = \operatorname{dn}(\tau)\left\{1-\left[1+2k^2 \pm \sqrt{1-k^2+4k^4}\right]\operatorname{sn}^2(\tau)\right\}$$
$$f_3 = \operatorname{sn}(\tau)\operatorname{cn}(\tau)\operatorname{dn}(\tau).$$

- $n = 4$

For $j = 1, 3', 5'$

$$h_j = \frac{20}{3}(1+k^2) + \frac{8\sqrt{13}}{3}\sqrt{1-k^2+k^4}\cos\frac{\vartheta_j}{3}$$

where $\vartheta_1 = \vartheta$, $\vartheta_{3'} = \vartheta + 4\pi$, $\vartheta_{5'} = \vartheta + 2\pi$ and

$$\cos\vartheta = \frac{35}{26\sqrt{13}}\frac{(1+k^2)(1-2k^2)(2-k^2)}{\sqrt{(1-k^2+k^4)^3}}$$
$$h_{2,4} = 5(2+k^2) \pm 2\sqrt{9-9k^2+4k^4}$$
$$h_{2',4'} = 5(1+2k^2) \pm 2\sqrt{4-9k^2+9k^4}$$
$$h_{3,5} = 5(1+k^2) \pm 2\sqrt{4+k^2+4k^4}$$
$$f_j = 1 - \frac{h_j}{2}\operatorname{sn}^2(\tau) + \left[\frac{5}{3}k^2 - \frac{1}{6}(1+k^2)h_j + \frac{1}{24}h_j^2\right]\operatorname{sn}^4(\tau) \qquad \text{for} \quad j = 1, 3', 5'$$
$$f_{2,4} = \operatorname{sn}(\tau)\operatorname{cn}(\tau)\left\{1-\frac{1}{3}\left[3+2k^2 \pm \sqrt{9-9k^2+4k^4}\right]\operatorname{sn}^2(\tau)\right\}$$
$$f_{2',4'} = \operatorname{sn}(\tau)\operatorname{dn}(\tau)\left\{1-\frac{1}{3}\left[2+3k^2 \pm \sqrt{4-9k^2+9k^4}\right]\operatorname{sn}^2(\tau)\right\}$$
$$f_{3,5} = \operatorname{cn}(\tau)\operatorname{dn}(\tau)\left\{1-\left[2+2k^2 \pm \sqrt{4+k^2+4k^4}\right]\operatorname{sn}^2(\tau)\right\}.$$

- $n = 5$

For $j = 1, 3', 5'$

$$h_j = \frac{35(1+k^2)}{3} + \frac{8}{3}\sqrt{28-43k^2+28k^4}\cos\frac{\vartheta_j}{3} \qquad \vartheta_{3'} = \vartheta_1 + 4\pi \qquad \vartheta_{5'} = \vartheta_1 + 2\pi$$

for $j = 2, 4, 6$

$$h_j = \frac{5(7+4k^2)}{3} + \frac{8}{3}\sqrt{28-13k^2+13k^4}\cos\frac{\vartheta_j}{3} \qquad \vartheta_4 = \vartheta_2 + 4\pi \qquad \vartheta_6 = \vartheta_2 + 2\pi$$

for $j = 2', 4', 6'$

$$h_j = \frac{5(4+7k^2)}{3} + \frac{8}{3}\sqrt{13-13k^2+28k^4}\cos\frac{\vartheta j}{3} \qquad \vartheta_{4'} = \vartheta_{2'} + 4\pi \qquad \vartheta_{6'} = \vartheta_{2'} + 2\pi$$
$$h_3 = 10(1+k^2) + 6\sqrt{1-k^2+k^4}$$
$$h_5 = 10(1+k^2) - 6\sqrt{1-k^2+k^4}$$

where

$$\cos\vartheta_1 = \frac{5(32 - 39k^2 - 39k^4 + 32k^6)}{2\sqrt{(28 - 43k^2 + 28k^4)^3}}$$

$$\cos\vartheta_2 = \frac{5(32 - 57k^2 - 21k^4 + 14k^6)}{2\sqrt{(28 - 13k^2 + 13k^4)^3}}$$

$$\cos\vartheta_{2'} = \frac{5(14 - 21k^2 - 57k^4 + 32k^6)}{2\sqrt{(13 - 13k^2 + 28k^4)^3}}$$

$$f_j = f(\tau)\{1 - a_j \,\mathrm{sn}^2(\tau) + b_j \,\mathrm{sn}^4(\tau)\}$$

where for $j = 1, 3', 5'$

$$f(\tau) = \mathrm{sn}(\tau) \qquad a_j = \tfrac{1}{6}\{h_j - (1 + k^2)\}$$

$$b_j = \tfrac{1}{5}\left\{7k^2 + \tfrac{3}{8}(1 + k^2)\right\} - \tfrac{1}{12}(1 + k^2)h_j + \tfrac{1}{120}h_j^2$$

for $j = 2, 4, 6$

$$f(\tau) = \mathrm{cn}(\tau) \qquad a_j = \tfrac{1}{2}(h_j - 1) \qquad b_j = \tfrac{1}{8}(3 + 20k^2) - \tfrac{1}{12}(5 + 2k^2)h_j + \tfrac{1}{24}h_j^2$$

for $j = 2', 4', 6'$

$$f(\tau) = \mathrm{dn}(\tau) \qquad a_j = \tfrac{1}{2}(h_j - k^2) \qquad b_j = \tfrac{1}{8}k^2(20 + 3k^2) - \tfrac{1}{12}(2 + 5k^2)h_j + \tfrac{1}{24}h_j^2$$

for $j = 3, 5$

$$f(\tau) = \mathrm{sn}(\tau)\,\mathrm{cn}(\tau)\,\mathrm{dn}(\tau) \qquad a_j = \tfrac{1}{6}\{h_j - 4(1 + k^2)\} \qquad b_j = 0.$$

Appendix B

In this appendix, we present specific analytic solutions of equation (3) involving specific combinations for interaction type $(+ + + \cdots +)$, $(- + - \cdots -)$ and $(+ - + - \cdots -)$ for $N = 1$–4 and $n = N$. These combinations are not the only ones possible for the given interaction types, as can be seen from appendix C, but they have the special feature of having the sn function (or the cn function which can be transformed into a constant multiple of sn/dn by displacing the argument by a quarter period) as a factor in every component. These special solutions are useful for the study of coupled Gross–Pitaevskii equations [8].

- $N = 1$
  - (i) Combination $(1)_1$ for interaction type $(-)$.

    $$\psi_1(t) = C_1 \,\mathrm{sn}(\alpha t)$$

    where $C_1 = \sqrt{2}k\alpha$, $c_1 = (1 + k^2)\alpha^2$.
  - (ii) Combination $(2)_1$ for interaction type $(+)$.

    $$\psi_1(t) = C_1 \,\mathrm{cn}(\alpha t)$$

    where $C_1 = \sqrt{2}k\alpha$, $c_1 = (1 - 2k^2)\alpha^2$.

- $N = 2$
  - (i) Combination $(2, 2')_2$ for interaction type $(+-)$.

    $$\psi_1(t) = C_1 \,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t) \qquad \psi_2(t) = C_2 \,\mathrm{sn}(\alpha t)\,\mathrm{dn}(\alpha t)$$

    where

    $$C_1 = \sqrt{6}k^2k'^{-1}\alpha \qquad C_2 = \sqrt{6}kk'^{-1}\alpha \qquad c_1 = (4 + k^2)\alpha^2 \qquad c_2 = (1 + 4k^2)\alpha^2.$$

(ii) Combination $(2, 3)_2$ for interaction type (++).

$$\psi_1(t) = C_1\,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t) \qquad \psi_2(t) = C_2\,\mathrm{cn}(\alpha t)\,\mathrm{dn}(\alpha t)$$

where

$$C_1 = \sqrt{6}k^2\alpha \qquad C_2 = \sqrt{6}k\alpha \qquad c_1 = (4 - 5k^2)\alpha^2 \qquad c_2 = (1 - 5k^2)\alpha^2.$$

- $N = 3$

(i) Combination $(1, 3, 3')_3$ for interaction type $(- + -)$.

$$\psi_{1,3}(t) = C_{1,3}\,\mathrm{sn}(\alpha t)\{1 - a_{1,3'}\,\mathrm{sn}^2(\alpha t)\} \qquad \psi_2(t) = C_2\,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t)\,\mathrm{dn}(\alpha t)$$

where

$$a_{1,3'} = \tfrac{1}{3}\left[2 + 2k^2 \pm \sqrt{4 - 7k^2 + 4k^4}\right]$$
$$C_{1,3} = \sqrt{3}k'^{-2}k\alpha\left(2 + k^2 + 2k^4 \mp \frac{(1 + k^2)(8 - 11k^2 + 8k^4)}{2\sqrt{4 - 7k^2 + 4k^4}}\right)^{1/2}$$
$$C_2 = \sqrt{30}k'^{-2}k\alpha$$
$$c_{1,3} = \left\{5(1 + k^2) \pm 2\sqrt{4 - 7k^2 + 4k^4}\right\}\alpha^2 \qquad c_2 = 4(1 + k^2)\alpha^2.$$

(ii) Combination $(2, 3, 4)_3$ for interaction type $(+ + +)$.

$$\psi_{1,3}(t) = C_{1,3}\,\mathrm{cn}(\alpha t)\{1 - a_{2,4}\,\mathrm{sn}^2(\alpha t)\} \qquad \psi_2(t) = C_2\,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t)\,\mathrm{dn}(\alpha t)$$

where

$$a_{2,4} = 2 + k^2 \pm \sqrt{4 - k^2 + k^4}$$
$$C_{1,3} = \sqrt{3}k\alpha\left(2 \mp \frac{8 - k^2}{2\sqrt{4 - k^2 + k^4}}\right)^{1/2} \qquad C_2 = \sqrt{30}k^2\alpha$$
$$c_{1,3} = \left\{5(1 - 2k^2) \pm 2\sqrt{4 - k^2 + k^4}\right\}\alpha^2 \qquad c_2 = 4(1 - 2k^2)\alpha^2.$$

- $N = 4$

(i) Combination $(2, 2', 4, 4')_4$ for interaction type $(+-+-)$.

$$\psi_{1,3}(t) = C_{1,3}\,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t)\{1 - b_{2,4}\,\mathrm{sn}^2(\alpha t)\}$$
$$\psi_{2,4}(t) = C_{2,4}\,\mathrm{sn}(\alpha t)\,\mathrm{dn}(\alpha t)\{1 - b_{2',4'}\,\mathrm{sn}^2(\alpha t)\}$$

where

$$b_{2,4} = \tfrac{1}{3}\left[3 + 2k^2 \pm \sqrt{9 - 9k^2 + 4k^4}\right] \qquad b_{2',4'} = \tfrac{1}{3}\left[2 + 3k^2 \pm \sqrt{4 - 9k^2 + 9k^4}\right]$$
$$C_{1,3} = \sqrt{5}k'^{-3}k^2\alpha\left(9 + 3k^2 + 2k^4 \mp \frac{54 - 9k^2 + 3k^4 + 8k^6}{2\sqrt{9 - 9k^2 + 4k^4}}\right)^{1/2}$$
$$C_{2,4} = \sqrt{5}k'^{-3}k\alpha\left(2 + 3k^2 + 9k^4 \mp \frac{8 + 3k^2 - 9k^4 + 54k^6}{2\sqrt{4 - 9k^2 + 9k^4}}\right)^{1/2}$$
$$c_{1,3} = \left\{5(2 + k^2) \pm 2\sqrt{9 - 9k^2 + 4k^4}\right\}\alpha^2$$
$$c_{2,4} = \left\{5(1 + 2k^2) \pm 2\sqrt{4 - 9k^2 + 9k^4}\right\}\alpha^2.$$

(ii) Combination $(2, 3, 4, 5)_4$ for interaction type $(+ + + +)$.

$$\psi_{1,3}(t) = C_{1,3}\,\mathrm{sn}(\alpha t)\,\mathrm{cn}(\alpha t)\{1 - b_{2,4}\,\mathrm{sn}^2(\alpha t)\}$$
$$\psi_{2,4}(t) = C_{2,4}\,\mathrm{cn}(\alpha t)\,\mathrm{dn}(\alpha t)\{1 - b_{3,5}\,\mathrm{sn}^2(\alpha t)\}$$

where

$$b_{3,5} = 2 + 2k^2 \pm \sqrt{4 + k^2 + 4k^4}$$
$$C_{1,3} = 3\sqrt{5}k^2\alpha\left(1 \mp \frac{3(2 - k^2)}{2\sqrt{9 - 9k^2 + 4k^4}}\right)^{1/2}$$

$$C_{2,4} = \sqrt{5}k\alpha\left(2 \mp \frac{8+k^2}{2\sqrt{4+k^2+4k^4}}\right)^{1/2}$$
$$c_{1,3} = \left\{5(2-3k^2) \pm 2\sqrt{9-9k^2+4k^4}\right\}\alpha^2$$
$$c_{2,4} = \left\{5(1-3k^2) \pm 2\sqrt{4+k^2+4k^4}\right\}\alpha^2.$$

Appendix C

A collection of N Lamé functions $\left(f_{m_1}^{(n)}, f_{m_2}^{(n)}, \ldots, f_{m_N}^{(n)}\right)$ or $(m_1, m_2, \ldots, m_N)_n$ chosen from $2n+1$ Lamé functions $f_m^{(n)}$ of order n that can serve as an analytic solution for the N components ψ_m, $m = 1, \ldots, N$, of equation (3) will be referred to as a combination. In this appendix, we list, for every one of the 2^N possible interaction types $(\beta_1, \beta_2, \ldots, \beta_N)$, where β_j can be +1 or -1, for equation (3), all the possible combinations for Lamé functions of order $n = N$, for $N =$ 1–5 (the subscript n in the combination $(m_1, m_2, \ldots, m_N)_n$ is dropped as it is understood that $n = N$. The total number M of possible combinations for $N = 1, 2, 3, 4, 5$ are 3, 10, 35, 126 and 462 respectively. These groupings, which have been checked analytically up to $N = 4$, and numerically up to $N = 5$, confirm rules (I) given in section 3. To obtain the corresponding amplitudes C_j and the appropriate c_m for equation (3) for each allowed combination, we need simply to substitute $\psi_j = C_j f_{m_j}^{(n)}$ into the equations, using the $f_m^{(n)}$ given in appendix A, and solve a set of simultaneous algebraic equations as prescribed in [4].

We list only the 'principal' combinations with the number of total possible combinations that can be obtained from them by changing, say, 2 to 2′, 3 to 3′, etc, given in the square parentheses that follow, remembering the restriction that for any combination $(m_1, m_2, \ldots, m_N)_n$ for the case $n = N, m_1 < m_2 < \cdots < m_N$. For example, (2)[2] represents two combinations $(2)_1$ and $(2')_1$, and $(1, 2, 3, 4, 4')$[4] represents four combinations $(1, 2, 3, 4, 4')_5$, $(1, 2, 3', 4, 4')_5$, $(1, 2', 3, 4, 4')_5$ and $(1, 2', 3', 4, 4')_5$.

Interaction type	*Combination*
$N = 1$	
$(-)$	(1)[1]
$(+)$	(2)[2]
$N = 2$	
$(--)$	(1, 2)[2]
$(-+)$	(1, 3)[2], (3, 3′)[1]
$(+-)$	(2, 2′)[1]
$(++)$	(2, 3)[4]
$N = 3$	
$(---)$	(1, 2, 3)[4]
$(--+)$	(1, 2, 2′)[1], (1, 2, 4)[4], (1, 4, 4′)[1], (3, 4, 4′)[2]
$(-+-)$	(1, 3, 3′)[1]
$(-++)$	(1, 3, 4)[4], (3, 3′, 4)[2]
$(+--)$	(2, 2′, 3)[2]
$(+-+)$	(2, 2′, 4)[2], (2, 4, 4′)[2]
$(++-)$	(2, 3, 3′)[2]
$(+++)$	(2, 3, 4)[8]

$N = 4$

$(- - - -)$	$(1, 2, 3, 4)[8]$
$(- - - +)$	$(1, 2, 3, 3')[2]$, $(1, 2, 3, 5)[8]$, $(1, 2, 5, 5')[2]$, $(1, 4, 5, 5')[2]$, $(3, 4, 5, 5')[4]$
$(- - + -)$	$(1, 2, 2', 4)[2]$, $(1, 2, 4, 4')[2]$
$(- - + +)$	$(1, 2, 2', 3)[2]$, $(1, 2, 2', 5)[2]$, $(1, 2, 4, 5)[8]$, $(1, 4, 4', 5)[2]$, $(3, 4, 4', 5)[4]$
$(- + - -)$	$(1, 3, 3', 4)[2]$
$(- + - +)$	$(1, 3, 3', 5)[2]$, $(1, 3, 5, 5')[2]$, $(3, 3', 5, 5')[1]$
$(- + + -)$	$(1, 3, 4, 4')[2]$, $(3, 3', 4, 4')[1]$
$(- + + +)$	$(1, 3, 4, 5)[8]$, $(3, 3', 4, 5)[4]$
$(+ - - -)$	$(2, 2', 3, 4)[4]$
$(+ - - +)$	$(2, 2', 3, 3')[1]$, $(2, 2', 3, 5)[4]$, $(2, 2', 5, 5')[1]$, $(2, 4, 5, 5')[4]$, $(4, 4', 5, 5')[1]$
$(+ - + -)$	$(2, 2', 4, 4')[1]$
$(+ - + +)$	$(2, 2', 4, 5)[4]$, $(2, 4, 4', 5)[4]$
$(+ + - -)$	$(2, 3, 3', 4)[4]$,
$(+ + - +)$	$(2, 3, 3', 5)[4]$, $(2, 3, 5, 5')[4]$
$(+ + + -)$	$(2, 3, 4, 4')[4]$
$(+ + + +)$	$(2, 3, 4, 5)[16]$

$N = 5$

$(- - - - -)$	$(1, 2, 3, 4, 5)[16]$
$(- - - - +)$	$(1, 2, 3, 4, 4')[4]$, $(1, 2, 3, 4, 6)[16]$, $(1, 2, 3, 6, 6')[4]$, $(1, 2, 5, 6, 6')[4]$, $(1, 4, 5, 6, 6')[4]$, $(3, 4, 5, 6, 6')[8]$
$(- - - + -)$	$(1, 2, 3, 3', 5)[4]$, $(1, 2, 3, 5, 5')[4]$
$(- - - + +)$	$(1, 2, 3, 3', 4)[4]$, $(1, 2, 3, 3', 6)[4]$, $(1, 2, 3, 5, 6)[16]$, $(1, 2, 5, 5', 6)[4]$, $(1, 4, 5, 5', 6)[4]$, $(3, 4, 5, 5', 6)[8]$
$(- - + - -)$	$(1, 2, 2', 4, 5)[4]$, $(1, 2, 4, 4', 5)[4]$
$(- - + - +)$	$(1, 2, 2', 4, 4')[1]$, $(1, 2, 2', 4, 6)[4]$, $(1, 2, 2', 6, 6')[1]$, $(1, 2, 4, 4', 6)[4]$, $(1, 2, 4, 6, 6')[4]$, $(1, 4, 4', 6, 6')[1]$, $(3, 4, 4', 6, 6')[2]$
$(- - + + -)$	$(1, 2, 2', 3, 3')[1]$, $(1, 2, 2', 3, 5)[4]$, $(1, 2, 2', 5, 5')[1]$, $(1, 2, 4, 5, 5')[4]$, $(1, 4, 4', 5, 5')[1]$, $(3, 4, 4', 5, 5')[2]$
$(- - + + +)$	$(1, 2, 2', 3, 4)[4]$, $(1, 2, 2', 3, 6)[4]$, $(1, 2, 2', 5, 6)[4]$, $(1, 2, 4, 5, 6)[16]$, $(1, 4, 4', 5, 6)[4]$, $(3, 4, 4', 5, 6)[8]$
$(- + - - -)$	$(1, 3, 3', 4, 5)[4]$
$(- + - - +)$	$(1, 3, 3', 4, 4')[1]$, $(1, 3, 3', 4, 6)[4]$, $(1, 3, 3', 6, 6')[1]$, $(1, 3, 5, 6, 6')[4]$, $(1, 5, 5', 6, 6')[1]$, $(3, 5, 5', 6, 6')[2]$, $(3, 3', 5, 6, 6')[2]$
$(- + - + -)$	$(1, 3, 3', 5, 5')[1]$
$(- + - + +)$	$(1, 3, 3', 5, 6)[4]$, $(1, 3, 5, 5', 6)[4]$, $(3, 3', 5, 5', 6)[2]$
$(- + + - -)$	$(1, 3, 4, 4', 5)[4]$, $(3, 3', 4, 4', 5)[2]$
$(- + + - +)$	$(1, 3, 4, 4', 6)[4]$, $(1, 3, 4, 6, 6')[4]$, $(3, 3', 4, 4', 6)[2]$, $(3, 3', 4, 6, 6')[2]$

$(-+++-)$	$(1,3,4,5,5')[4]$, $(3,3',4,5,5')[2]$
$(-++++)$	$(1,3,4,5,6)[16]$, $(3,3',4,5,6)[8]$
$(+----)$	$(2,2',3,4,5)[8]$
$(+---+)$	$(2,2',3,4,4')[2]$, $(2,2',3,4,6)[8]$, $(2,2',3,6,6')[2]$, $(2,2',5,6,6')[2]$, $(2,4,5,6,6')[8]$, $(4,4',5,6,6')[2]$
$(+--+-)$	$(2,2',3,3',5)[2]$, $(2,2',3,5,5')[2]$
$(+--++)$	$(2,2',3,3',4)[2]$, $(2,2',3,3',6)[2]$, $(2,2',3,5,6)[8]$, $(2,2',5,5',6)[2]$, $(2,4,5,5',6)[8]$, $(4,4',5,5',6)[2]$
$(+-+--)$	$(2,2',4,4',5)[2]$
$(+-+-+)$	$(2,2',4,4',6)[2]$, $(2,2',4,6,6')[2]$, $(2,4,4',6,6')[2]$
$(+-++-)$	$(2,2',4,5,5')[2]$, $(2,4,4',5,5')[2]$
$(+-+++)$	$(2,2',4,5,6)[8]$, $(2,4,4',5,6)[8]$
$(++---)$	$(2,3,3',4,5)[8]$
$(++--+)$	$(2,3,3',4,4')[2]$, $(2,3,3',4,6)[8]$, $(2,3,3',6,6')[2]$, $(2,3,5,6,6')[8]$, $(2,5,5',6,6')[2]$, $(4,5,5',6,6')[2]$
$(++-+-)$	$(2,3,3',5,5')[2]$
$(++-++)$	$(2,3,3',5,6)[8]$, $(2,3,5,5',6)[8]$
$(+++--)$	$(2,3,4,4',5)[8]$
$(+++-+)$	$(2,3,4,4',6)[8]$, $(2,3,4,6,6')[8]$
$(++++-)$	$(2,3,4,5,5')[8]$
$(+++++)$	$(2,3,4,5,6)[32]$

Appendix D

In this appendix, we give specific examples of rule (II) given in section 3 for the disallowed combinations of Lamé functions of order $n < N$ for N CNLS equations, for $N = 2$ and 3 only, as those for higher values of N can be obtained using rule (II) without too much trouble. The numbers given in the square brackets have the same meaning as those in appendix C, remembering, however, that in the case of $n < N$, some or all of the m in any combination $(m_1, m_2, \ldots, m_N)_n$ can be equal, with the restriction only that $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_N$. For example, $(2,2)_1[3]$ represents three combinations $(2,2)_1$, $(2,2')_1$, $(2',2')_1$, and $(1,2,3)_2[4]$ represents four combinations $(1,2,3)_2$, $(1,2,3')_2$, $(1,2',3)_2$, $(1,2',3')_2$.

Interaction type	*Disallowed combination*
$N = 2$	
$(--)$	$(2,2)_1[3]$
$(-+)$	*None*
$(+-)$	$(1,2)_1[2]$
$(++)$	$(1,1)_1[1]$
$N = 3$	
$(---)$	$(2,2,2)_1[4]$, $(1,1,1)_2[1]$, $(1,1,3)_2[2]$, $(1,3,3)_2[3]$, $(3,3,3)_2[4]$
$(--+)$	$(1,1,2)_2[2]$
$(-+-)$	$(1,2,3)_2[4]$
$(-++)$	$(1,2,2)_2[3]$
$(+--)$	$(1,2,2)_1[3]$, $(2,3,3)_2[6]$

$(+-+)$	*None*
$(++-)$	$(1,1,2)_1[2]$, $(2,2,3)_2[6]$
$(+++)$	$(1,1,1)_1[1]$, $(2,2,2)_2[4]$.

Appendix E

In this appendix, we illustrate how the C_m and c_m for equations (3) and (8) for a solution given by a degenerate combination $(m_1, \ldots, m_N)_n$ in which some of the m are equal, can be expressed in terms of the corresponding values of the solution given by the contracted (non-degenerate) combination.

Consider the case in which only two of the m, say, $m_p = m_{p+1}$ in

$$(m_1, \ldots, m_p, m_{p+1}, \ldots, m_N)_n \qquad n < N \tag{E.1}$$

as the generalization will become obvious. We exclude m_{p+1} and consider the contracted non-degenerate combination

$$(m_1, \ldots, m_p, \ldots, m_N)_n \tag{E.2}$$

involving N' ($=N-1$ for this example) distinct Lamé functions of order n.

Suppose that we want the C_m and c_m for an allowed combination (E.1). Let the C and c for the combination (E.2) for the interaction type

$$((-1)^{s_1}, \ldots, (-1)^{s'_p}, (-1)^{s_{p+2}}, \ldots, (-1)^{s_N}) \tag{E.3}$$

be given by $C_j^{(n)}$ and $c_j^{(n)}$, $j = 1, \ldots, N'$. Then the C_m and c_m for combination (E.1) for interaction type $((-1)^{s_1}, \ldots, (-1)^{s_p}, (-1)^{s_{p+1}}, (-1)^{s_{p+2}}, \ldots, (-1)^{s_N})$ are given by

$$\begin{array}{lll} \text{for} \quad j = 1, \ldots, p-1 & C_j = C_j^{(n)} & c_j = c_j^{(n)} \\ \text{for} \quad j = p+2, \ldots, N & C_j = C_{j-1}^{(n)} & c_j = c_{j-1}^{(n)} \end{array}$$

and for $j = p$ and $j = p+1$

$$\begin{aligned} &(-1)^{s_p}(-1)^{m_p}(C_p)^2 + (-1)^{s_{p+1}}(-1)^{m_p}(C_{p+1})^2 = \left(C_p^{(n)}\right)^2 \\ &c_p = c_{p+1} = c_p^{(n)} \end{aligned} \tag{E.4}$$

where s_p and s_{p+1} can be arbitrary so long as (E.4) holds.

We give the following example of a degenerate combination for $N = 4, n = 2$, for which we want to find the appropriate c and C for equations (3) and (8) for this degenerate combination to be a solution for certain interaction types, using the c and C given in appendix B for the corresponding contracted non-degenerate combination and the formulae given above. We write the various c and C for various n in appendix B as $c^{(n)}$ and $C^{(n)}$.

Combination $(2, 2, 2, 2')_2$ *for interaction type* $(\beta_1, \beta_2, \beta_3, -)$. The required C and c are given in terms of $C_1^{(2)}, C_2^{(2)}$ and $c_1^{(2)}, c_2^{(2)}$ for the contracted non-degenerate combination $(2, 2')_2$ for interaction type $(+-)$ given in appendix B by

$$\begin{array}{ll} \beta_1 C_1^2 + \beta_2 C_2^2 + \beta_3 C_3^2 = \left\{C_1^{(2)}\right\}^2 & C_4 = C_2^{(2)} \\ c_1 = c_2 = c_3 = c_1^{(2)} & c_4 = c_2^{(2)} \end{array}$$

from which we see that C_1, C_2 and C_3 can be arbitrary but must satisfy the equation given above, and that β_1, β_2 and β_3 can take on $+1$ or -1 but have to exclude the case $\beta_1 = \beta_2 = \beta_3 = -1$. Thus interaction type $(----)$ is disallowed. In addition, from Section 3(II)(B), interaction type $(++++)$ is disallowed for this combination.

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